MATH2040 Linear Algebra II

Tutorial 6

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1 Examples:

Example 1

Prove the following *parallelogram law* on an inner product space V:

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2 \quad \forall x, y \in V.$$

Solution

Note

$$\begin{aligned} ||x+y||^2 &=< x+y, x+y >= ||x||^2 + < x, y > + < y, x > + ||y||^2 \quad \text{and} \\ ||x-y||^2 &=< x-y, x-y >= ||x||^2 - < x, y > - < y, x > + ||y||^2 \end{aligned}$$

So

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

Example 2

Suppose that $< \cdot, \cdot >_1$ and $< \cdot, \cdot >_2$ are two inner products on a vector space V. Prove that $< \cdot, \cdot >=< \cdot, \cdot >_1 + < \cdot, \cdot >_2$ is another inner product on V.

Solution

We need to check the following three conditions: Let $x, y, z \in V$ and $c \in \mathbb{F}$

1. (Linearity in the first argument)

$$\begin{aligned} < x+cy, z > &=< x+cy, z >_1 + < x+cy, z >_2 \\ &=< x, z >_1 + < x, z >_2 + c < y, z >_1 + c < y, z >_2 \\ &=< x, z > + c < y, z > \end{aligned}$$

2. (Conjugate symmetry)

$$\overline{< x, y >} = \overline{< x, y >_1} + \overline{< x, y >_2} =< y, x >_1 + < y, x >_2 =< y, x >$$

3. (Positivity)

$$< x, x >=< x, x >_1 + < x, x >_2 > 0 \quad \text{if } x \neq 0$$

Example 3

Let V be an inner product space over $\mathbb{C}$. Prove that for any $x, y \in V$:

$$< x, y >= \frac{1}{4}\sum_{k=1}^{4} i^k ||x + i^k y||^2.$$

Solution

Note

$$\begin{aligned}||x + i^k y||^2 &= ||x||^2 + 2Re(< x, i^k y >) + |i^k|^2 ||y||^2 \\ &= ||x||^2 + 2Re(\overline{i^k} < x, y >) + ||y||^2.\end{aligned}$$

Then

$$\begin{aligned}\sum_{k=1}^{4} i^k ||x + i^k y||^2 &= \sum_{k=1}^{4} i^k (||x||^2 + 2Re(\overline{i^k} < x, y >) + ||y||^2) \\ &= \sum_{k=1}^{4} i^k ||x||^2 + 2 \sum_{k=1}^{4} i^k Re(\overline{i^k} < x, y >) + \sum_{k=1}^{4} i^k ||y||^2 \\ &= ||x||^2 \sum_{k=1}^{4} (i^k) + 2 \sum_{k=1}^{4} i^k Re(\overline{i^k} < x, y >) + ||y||^2 \sum_{k=1}^{4} (i^k) \\ &= 2 \sum_{k=1}^{4} i^k Re(\overline{i^k} < x, y >)\end{aligned}$$

Let $< x, y >= a + bi$, so

$$\begin{aligned}\sum_{k=1}^{4} i^k ||x + i^k y||^2 &= 2 \sum_{k=1}^{4} [i^k Re(\overline{i^k} < x, y >)] \\ &= 2[i(b) - (-a) - i(-b) + (a)] \\ &= 4(a + bi) \\ &= 4 < x, y >\end{aligned}$$

2 Exercises:

Question 1 (Section 6.1 Q12):

Let $\{v_1, v_2, \ldots, v_n\}$ be an orthogonal set in an inner product space V, and let $a_1, a_2, \ldots, a_n$ be scalars. Prove that

$$\left|\left|\sum_{i=1}^{n} a_i v_i\right|\right|^2 = \sum_{i=1}^{n} |a_i|^2 ||v_i||^2.$$

Question 2 (Section 6.1 Q17):

Let T be a linear operator on an inner product space V,and suppose that $||T(x)|| = ||x||$ for all x. Prove that T is one-to-one.

Question 3 (Section 6.1 Q18):

Let V be a vector space over $\mathbb{F}$, and let W be an inner product space over $\mathbb{F}$ with inner product $< \cdot, \cdot >$. If $T : V \to W$ is linear, prove that $< x, y >' =< T(x), T(y) >$ defines an inner product on V if and only if T is one-to-one.

Question 4 (Section 6.1 Q20(a)):

Let V be an inner product space over $\mathbb{R}$, prove that for all $x, y \in V$

$$< x, y >= \frac{1}{4}||x + y||^2 - \frac{1}{4}||x - y||^2.$$

Solution

(Please refer to the practice problem set 6.)